Algebraic geometry 2

Exercise Sheet 4

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Exercise 1. Let $\varphi : A \to B$ be a homomorphism of rings and let $(f, f^\sharp) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ be the respective morphism of affine schemes.
(1) Let $h \in A$. Show that $f^{-1}(D(h)) = D(\varphi(h))$.
(2) Describe the ring homomorphism $f^\sharp_{D(h)} : \mathcal{O}_{\operatorname{Spec} A}(D(h)) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(D(h)))$.

Exercise 2. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X and let $\mathcal{B}$ be a basis of the topology on X. Show that φ is an isomorphism if and only if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for every open $U \in \mathcal{B}$.

Exercise 3. (1) Let A be a ring and $S \subset A$ a multiplicatively closed subset. Recall the description of prime ideals in $S^{-1}A$.
(2) Let $g \in A$ and let $\varphi : A \to A_g, a \mapsto \frac{a}{1}$, be the canonical map. Denote by $f : \operatorname{Spec} A_g \to \operatorname{Spec} A$ the respective map of topological spaces. Show that $\operatorname{Im} f = D(g)$ and moreover f induces a homemorphism $\operatorname{Spec} A_g \simeq D(g)$.
(3) Denote by $\mathcal{O}$ the structure sheaf on $\operatorname{Spec} A$. Show that the locally ringed space $(D(g), \mathcal{O}_{|D(g)})$ is isomorphic (as a locally ringed space) to $(\operatorname{Spec} A_g, \mathcal{O}_{\operatorname{Spec} A_g})$.
Hint: One may use Exercises 1 and 2.

Exercise 4. Let X be a scheme and A a ring. Show that the canonical map

$$\operatorname{Hom}_{\text{schemes}}(X, \operatorname{Spec} A) \to \operatorname{Hom}_{\text{rings}}(A, \mathcal{O}_X(X)), (f, f^\sharp) \mapsto f^\sharp_Y\,,$$

where $Y = \operatorname{Spec} A$, is a bijection.